

# Multiple linear regression

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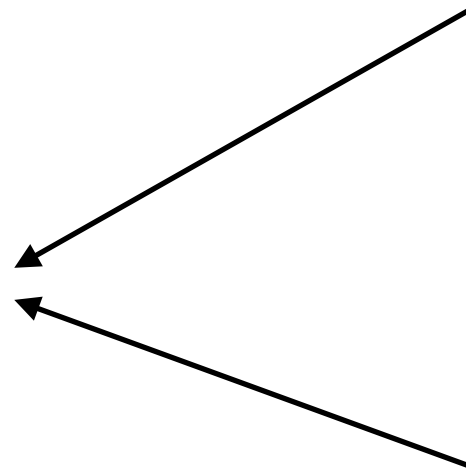
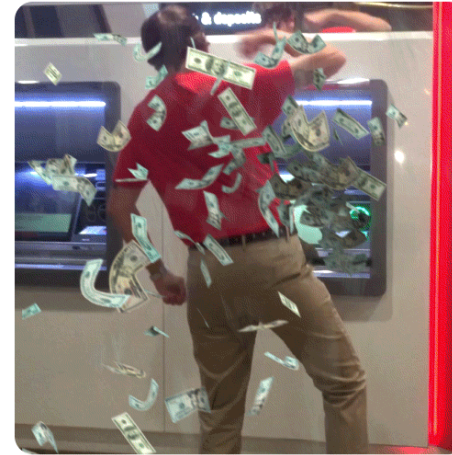
# Prologue:

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## Feedback and exercises

- None of you filled out the feedback survey....

# Introduction



- Build upon the example from previous session: what are the determinants of beer consumption?
  - We considered two variables **separately**: income and beer price
- Multiple regression analysis allows us to consider **both variables at once**
  - This changes the interpretation of the obtained estimates
  - They now give the association with the outcome variable, assuming that all other variables are held constant

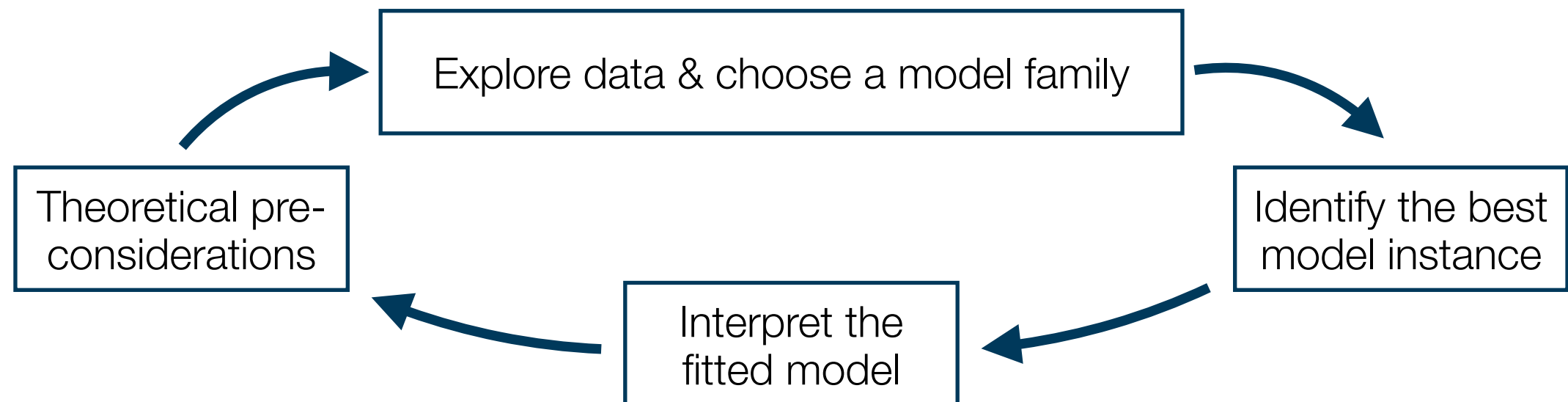
# Goals for today

- I. Learn how to implement and interpret multiple linear regression models
- II. Learn how to deal with categorical variables within a regression
- III. Understand the concept of interaction effect and the difference between interaction and parallel slopes models

# Multiple linear Regression

# Introduction

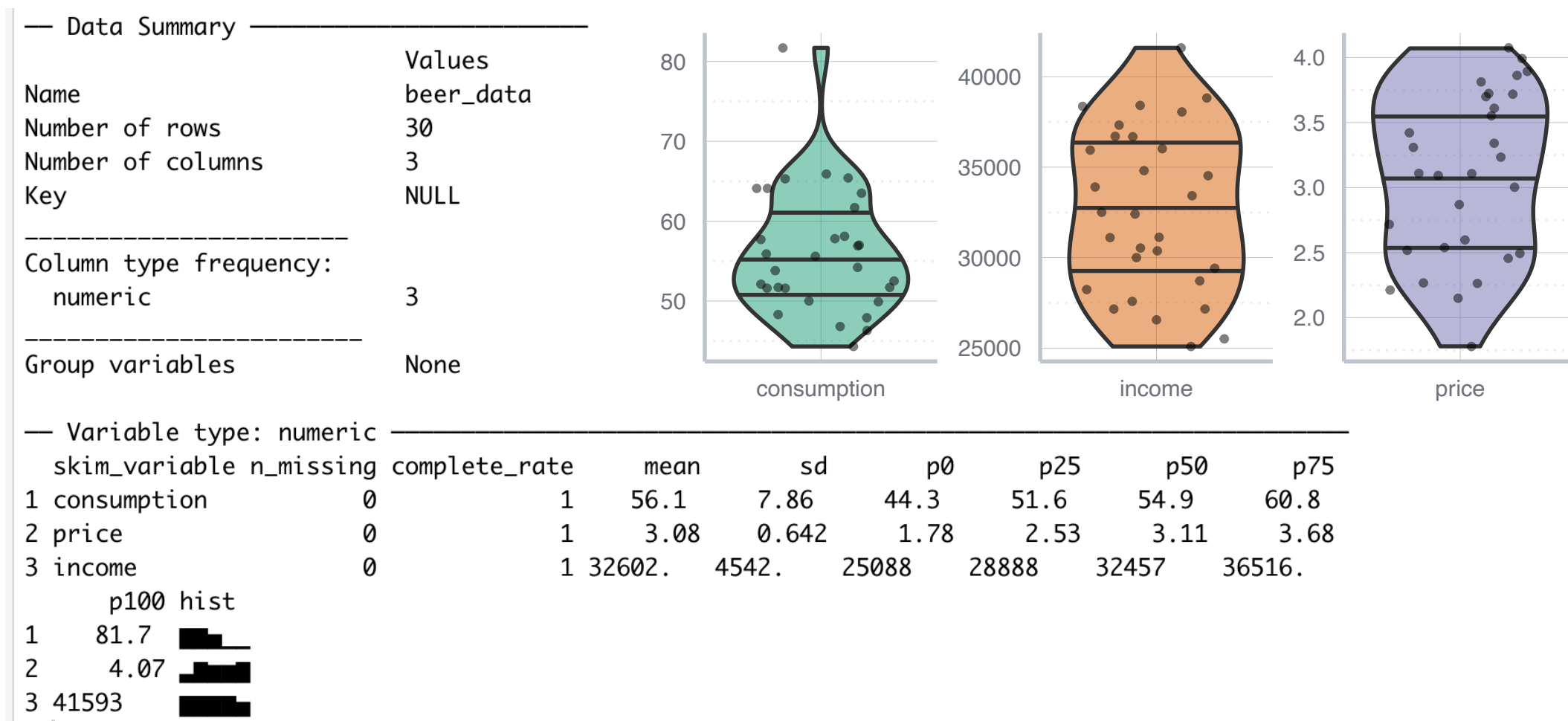
- In terms of theoretical background and technical implementation, multiple regression analysis is very similar to simple linear regression
- The overall sequence of considerations remains the same:



- Let us take this opportunity to recap what we have learned

# Data exploration

- We again use the data set `DataScienceExercises::beer`, but only the three variables of interest



- Since `consumption`, `income` and `price` are all numerical, we can basically proceed as in the previous session



# Data exploration

- Our focus on both income and price can be justified **theoretically** via reference to economic theory....
- ...and **empirically** by looking at the correlations:

```
> cor(beer_data$consumption, beer_data$price)
```

```
[1] -0.8038513
```

```
> cor(beer_data$consumption, beer_data$income)
```

```
[1] -0.714995
```

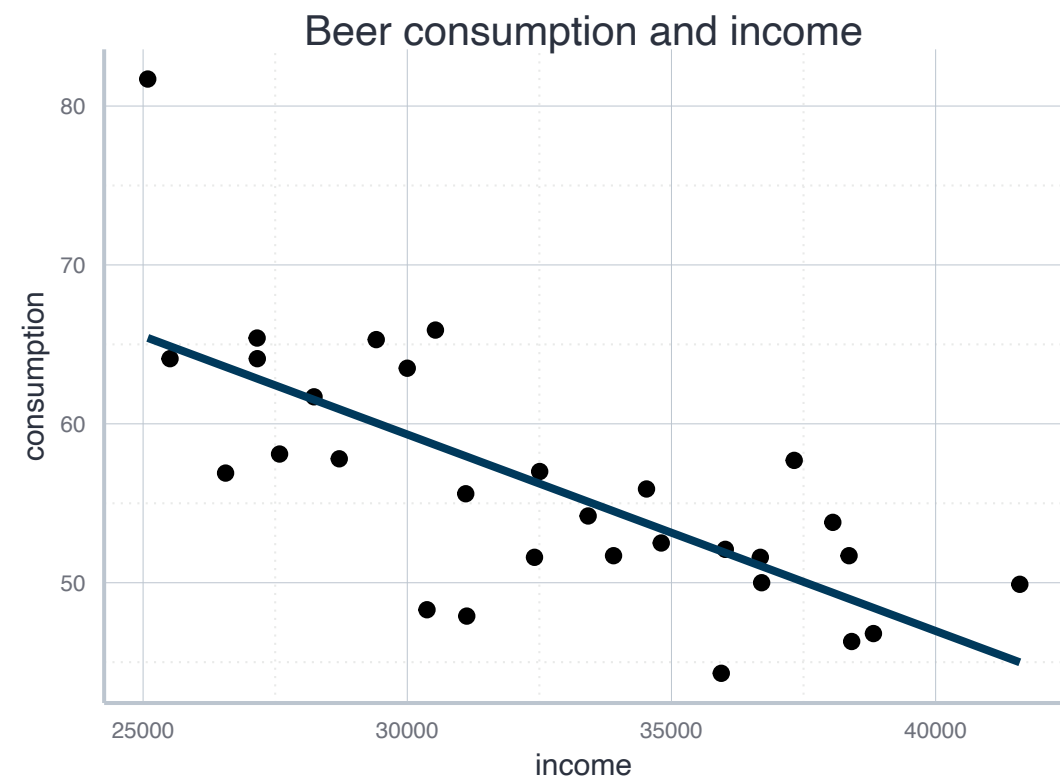
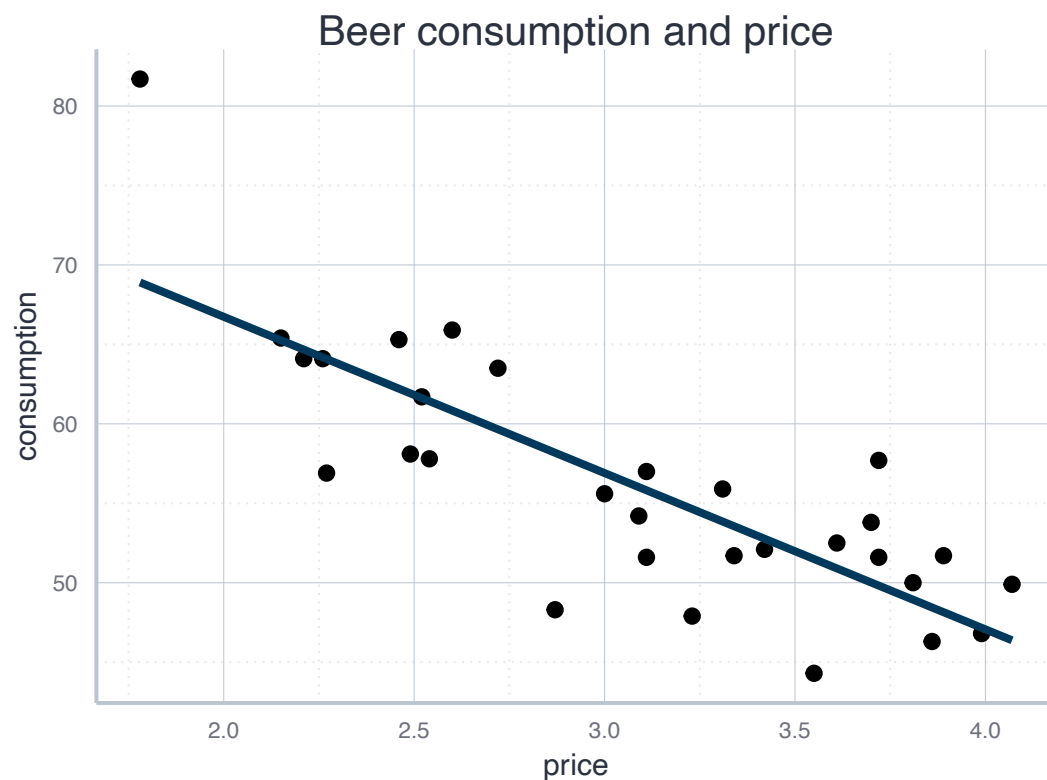
```
> cor(beer_data)
```

	consumption	price	income
consumption	1.0000000	-0.8038513	-0.7149950
price	-0.8038513	1.0000000	0.9763155
income	-0.7149950	0.9763155	1.0000000

Note: very strong correlations between explanatory variables should be a warning sign! More on this later!

# Data exploration

- Our focus on both income and price can be justified theoretically via reference to economic theory....
- ...and empirically by looking at the correlations:



➡ In both cases, a linear models seems to be an adequate choice!

# Estimate a multiple regression model

- Writing down our regression model with two explanatory variables is very similar to the case with only one variable:

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \epsilon$$

$$CONS = \beta_0 + \beta_1 PRICE + \beta_2 INCOME + \epsilon$$

- The computation in R is equally similar → here is the general form:

`lm(y ~ x1 + x2, data=data_used)`

- **Exercise:** adjust the code to the actual data set and estimate the model!

# Interpret a multiple regression model

```
> cons_model <- lm(consumption ~ price + income, data = beer_data)
> moderndive::get_regression_table(cons_model)
```

```
# A tibble: 3 × 7
```

	term	estimate	std_error	statistic	p_value	lower_ci	upper_ci
	<chr>	<dbl>	<dbl>	<dbl>	<dbl>	<dbl>	<dbl>
1	intercept	57.2	9.47	6.04	0	37.7	76.6
2	price	-27.7	5.44	-5.08	0	-38.8	-16.5
3	income	0.003	0.001	3.36	0.002	0.001	0.004

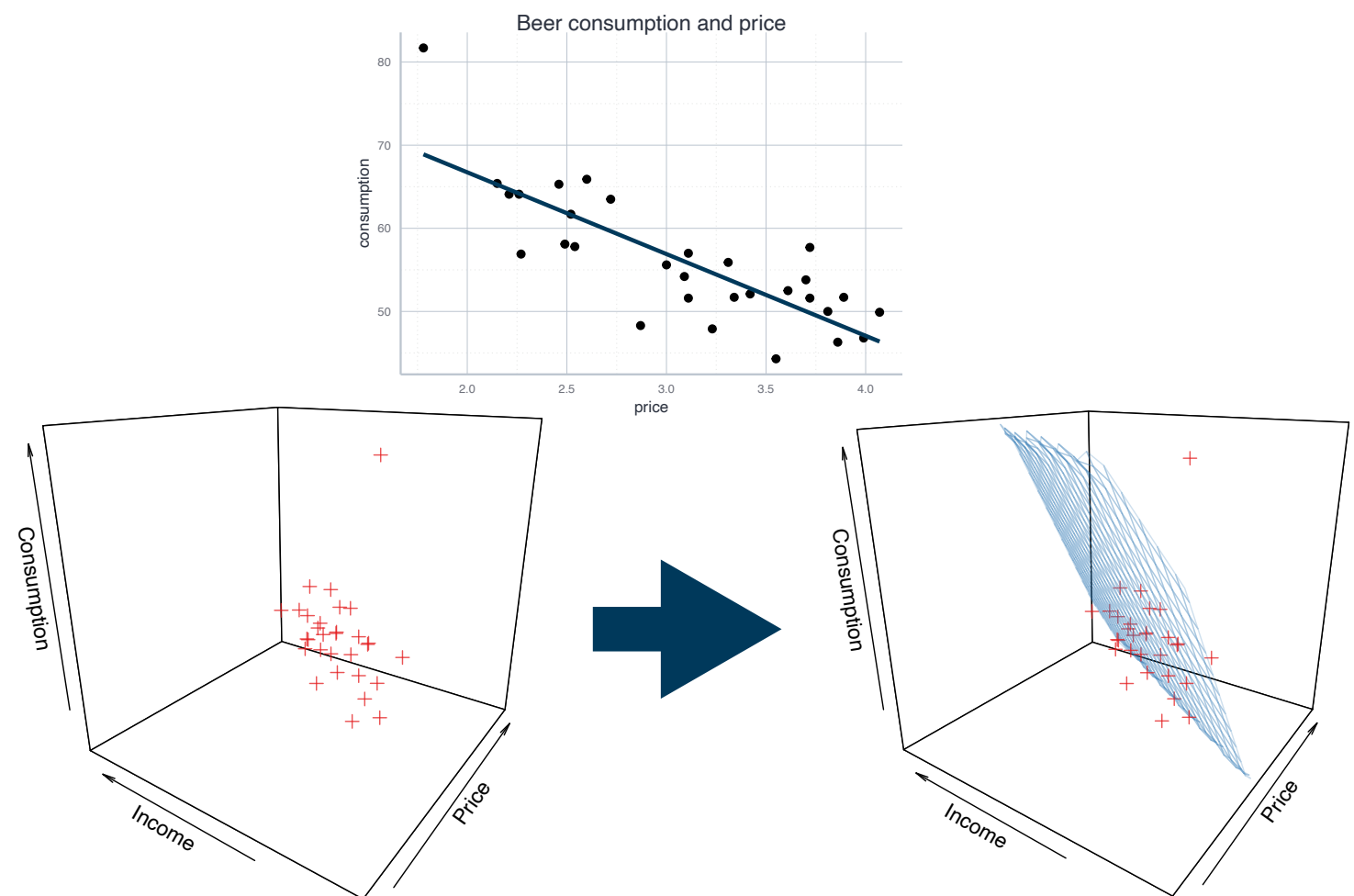
- In the multiple case, the coefficients must be interpreted in a *ceteris paribus* fashion:

For every increase of 1 unit in **price**, there is an associated decrease of, on average **and ceteris paribus**, 27.7 units of **consumption**.

For every increase of 1 unit in **income**, there is an associated decrease of, on average **and ceteris paribus**, 0.003 units of **consumption**.

# Graphical interpretation

- The more explanatory variables you use, the more difficult it becomes to think about the regression problem graphically
- In the simple regression case we fitted a regression line
- In the case of two explanatory variables we fit a regression plane
- In cases with more than two variables we fit a regression hyperplane
- But this is not easy to get your head around → if anything plot conditional relationships (see optional tutorial on the course page)



# Outlook: the choice of variables matters

- **Guess:** how do the estimates for income and price from the simple regression models and the multiple regression model relate to each other?

	Model 1	Model 2	Model 3
(Intercept)	86.406 *** (4.324)	96.439 *** (7.521)	57.160 *** (9.468)
price	-9.835 *** (1.375)		-27.653 *** (5.438)
income		-0.001 *** (0.000)	0.003 ** (0.001)
R <sup>2</sup>	0.646	0.511	0.750
Adj. R <sup>2</sup>	0.634	0.494	0.732
Num. obs.	30	30	30

- This points to an important concept: **omitted variable bias**
  - When you forget one important variable in your model, all resulting estimates can be misleading → more on this in later sessions

# Exercise

- Get together in groups and use again the data on beer consumption
- But this time use all potential explanatory variables for the RHS:
  - `price`: the price for beer
  - `price_liquor`: the price for other strong alcoholic beverages
  - `price_other`: price of other goods and services
  - `income`: household income
- Before you do the estimation, what would you expect regarding their effect?
- How can you interpret the estimates you obtained? How did the estimates change over different specifications?
- What specification would you prefer? Why?

# Categorical variables: Simple regression



# Using categorical variables

- So far we only worked with numerical and continuous variables
  - Income, prices, consumption,...
- But there are other types of variables, e.g. categorical data
  - Gender, continent of origin, employment status,...
- In the following we want to learn how to consider categorical data as **explanatory variables**
  - If you have categorical variables on the LHS → different estimation methods
- Let us illustrate the procedure using the data on life expectancy, but focus on the role of different continents
  - Data: `DataScienceExercises::gdplifexp2007`
  - Variables of interest: `continent`, `lifeExp`, and `gdpPercap`

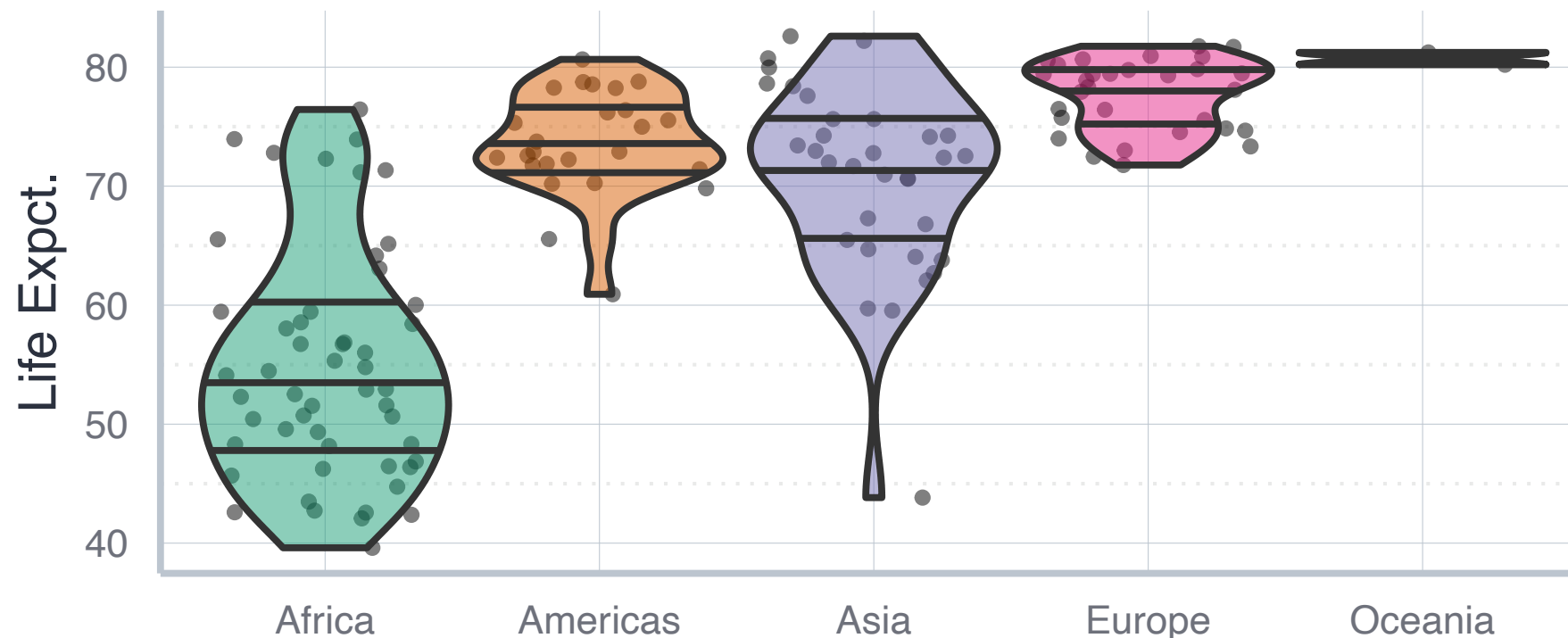
# Exploratory analysis

Data Summary		
Name	life_exp	Values
Number of rows	142	
Number of columns	3	
Key	NULL	
-----		
Column type frequency:		
factor	1	
numeric	2	
-----		
Group variables	None	
<div>No problems with missing data, 142 observations in total</div> <div>There are five different continents with Africa comprising most countries (52)</div> <div>Mean and median differ considerable due to skewed distribution of the variables!</div>		
Variable type: factor		
skim_variable	n_missing	complete_rate
1 continent	0	1
ordered	FALSE	n_unique top_counts
		5 Afr: 52, Asi: 33, Eur: 30, Ame: 25
Variable type: numeric		
skim_variable	n_missing	complete_rate
1 lifeExp	0	1
2 gdpPercap	0	1
	mean	sd
	p0	p25
	p50	p75
	p100	hist
	67.0	12.1
	39.6	57.2
	71.9	76.4
	82.6	
	11680.	12860.
	278.	1625.
	6124.	18009.
	49357.	

- Note: continent was saved as character, but we transformed it into factor

# Exploratory analysis

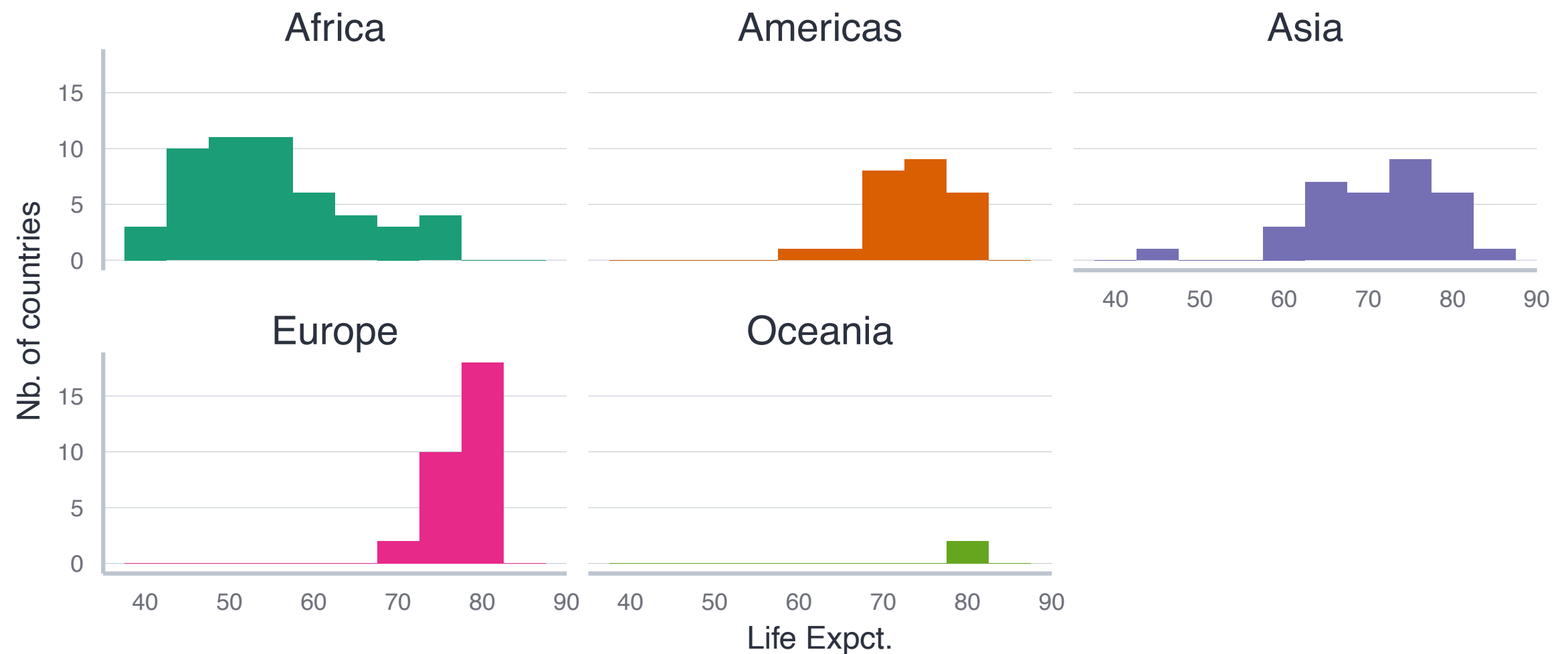
- We see considerable differences also within continents:



- Especially Oceania will be hard to interpret since it comprises only two countries

# Exploratory analysis

- To look at the distribution within countries, histograms are also useful:



- For categorical variables, fitting a regression line has a different meaning

# Fitting a model with categorical variables

- The notation for a model with a categorical variable on the RHS is similar...
  - ...but the technical implementation is quite different
- While we write:

$$lifeExp = \beta_0 + \beta_1 \cdot CONT + \epsilon$$

- What actually being estimated is:

$$lifeExp = \beta_0 + \beta_{Am.} \cdot \mathbb{I}_{Am.} CONT + \beta_{As.} \cdot \mathbb{I}_{As.} CONT + \beta_{Eu.} \cdot \mathbb{I}_{Eu.} CONT + \beta_{Oc.} \cdot \mathbb{I}_{Oc.} CONT + \epsilon$$

- Here  $\mathbb{I}_x(X)$  is an **indicator function** that takes the value 1 if  $X = x$  and zero otherwise
  - Thus  $\mathbb{I}_{Am.}(CONT) = 1$  iff  $CONT$  equals  $Am.$  (i.e. Americas), and 0 otherwise
  - Note that there are four indicator functions  $\rightarrow$  four continents (plus one as a baseline level)
- The estimates must, therefore, always be interpreted against a baseline value (here: the first factor level, i.e. Africa)

# Interpreting a model with categorical variables

$$lifeExp = \beta_0 + \beta_{Am.} \cdot \mathbb{I}_{Am.} CONT + \beta_{As.} \cdot \mathbb{I}_{As.} CONT + \beta_{Eu.} \cdot \mathbb{I}_{Eu.} CONT + \beta_{Oc.} \cdot \mathbb{I}_{Oc.} CONT + \epsilon$$

- Lets consider the results from estimating this formula one by one:
  - Note that the code for the regression remains `lm(lifeExp~continent)`

```
> cont_linmod <- lm(lifeExp~continent, data = life_exp)
```

```
> get_regression_table(cont_linmod)
```

```
# A tibble: 5 × 7
```

	term	estimate
	<chr>	<dbl>
1	intercept	54.8
2	continent: Americas	18.8
3	continent: Asia	15.9
4	continent: Europe	22.8
5	continent: Oceania	25.9

	continent	lifeExp_mean	diff_africa
	<fct>	<dbl>	<dbl>
1	Africa	54.8	0
2	Americas	73.6	18.8
3	Asia	70.7	15.9
4	Europe	77.6	22.8
5	Oceania	80.7	25.9

- The intercept corresponds to the mean value of the baseline category
  - The other estimates correspond to the deviation of the group mean from this baseline

# Interpreting a model with categorical variables

$$lifeExp = \beta_0 + \beta_{Am.} \cdot \mathbb{I}_{Am.} CONT + \beta_{As.} \cdot \mathbb{I}_{As.} CONT + \beta_{Eu.} \cdot \mathbb{I}_{Eu.} CONT + \beta_{Oc.} \cdot \mathbb{I}_{Oc.} CONT + \epsilon$$

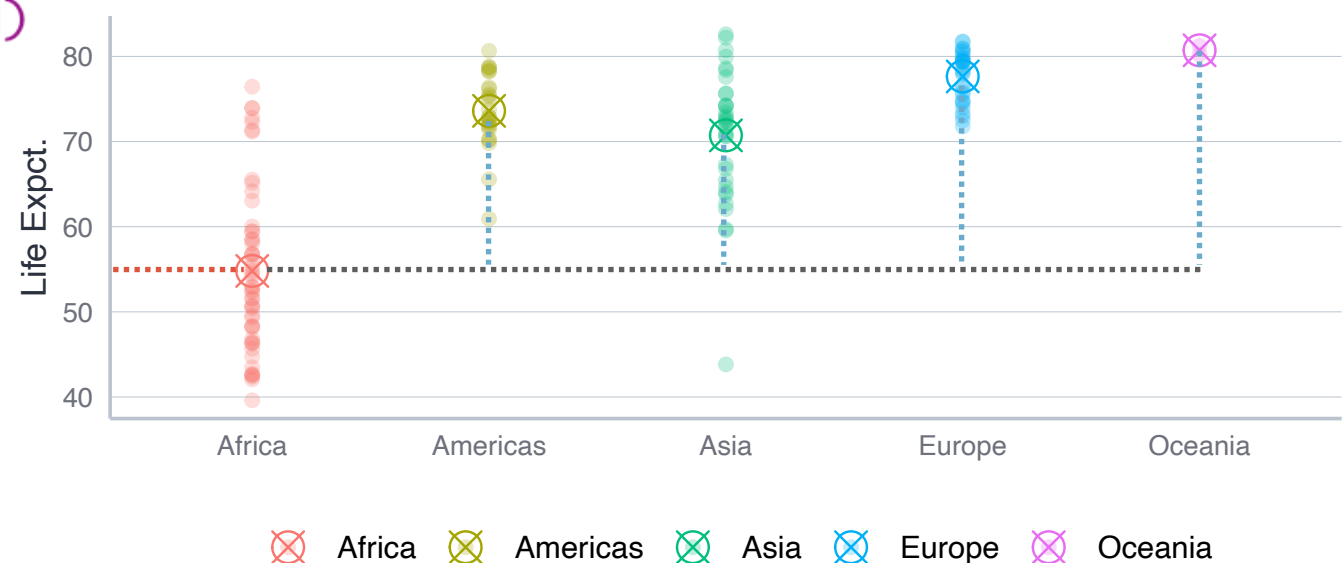
- Lets consider the results from estimating this formula one by one:
  - Note that the code for the regression remains `lm(lifeExp~continent)`

```
> cont_linmod <- lm(lifeExp~continent, data = life_exp)
```

```
> get_regression_table(cont_linmod)
```

```
# A tibble: 5 × 7
```

	term	estimate
	<chr>	<dbl>
1	intercept	54.8
2	continent: Americas	18.8
3	continent: Asia	15.9
4	continent: Europe	22.8
5	continent: Oceania	25.9



- The intercept corresponds to the mean value of the baseline category
  - The other estimates correspond to the deviation of the group mean from this baseline

# Quick recap

- The result of the following regression model...

$$SUGAR = \beta_0 + \beta_1 KIND + \epsilon$$

```
lm(`residual sugar` ~ kind, data = wine_data)
```

- ...is as follows:

term	estimate
<chr>	<dbl>
intercept	2.54
kind: white	3.85

- The variables are as follows:

- ``residual sugar``: the amount of sugar left in the wine
- `kind`: the kind of wine, red or white

- How would you interpret the estimated coefficients?



# Categorical variables: Multiple regression

# Introduction

- What if we would like to consider both continuous and categorical variables?
- In this case we must distinguish two cases: an **interaction model**, and a **parallel slope model**
  - Note: both also occur in the case without categorical variables, but here the distinction is most intuitive
- To illustrate the difference, we consider a data set on the prices of economics journals: `DataScienceExercises::econjournals`
  - Only consider journals that published at least 10 papers and cost under 5000 USD per year: `dplyr::filter(papers>10, sub_price<5000)`
- **Main interest:** what is the impact of the paper length on the subscription price? Are there differences between profit and nonprofit publishers?

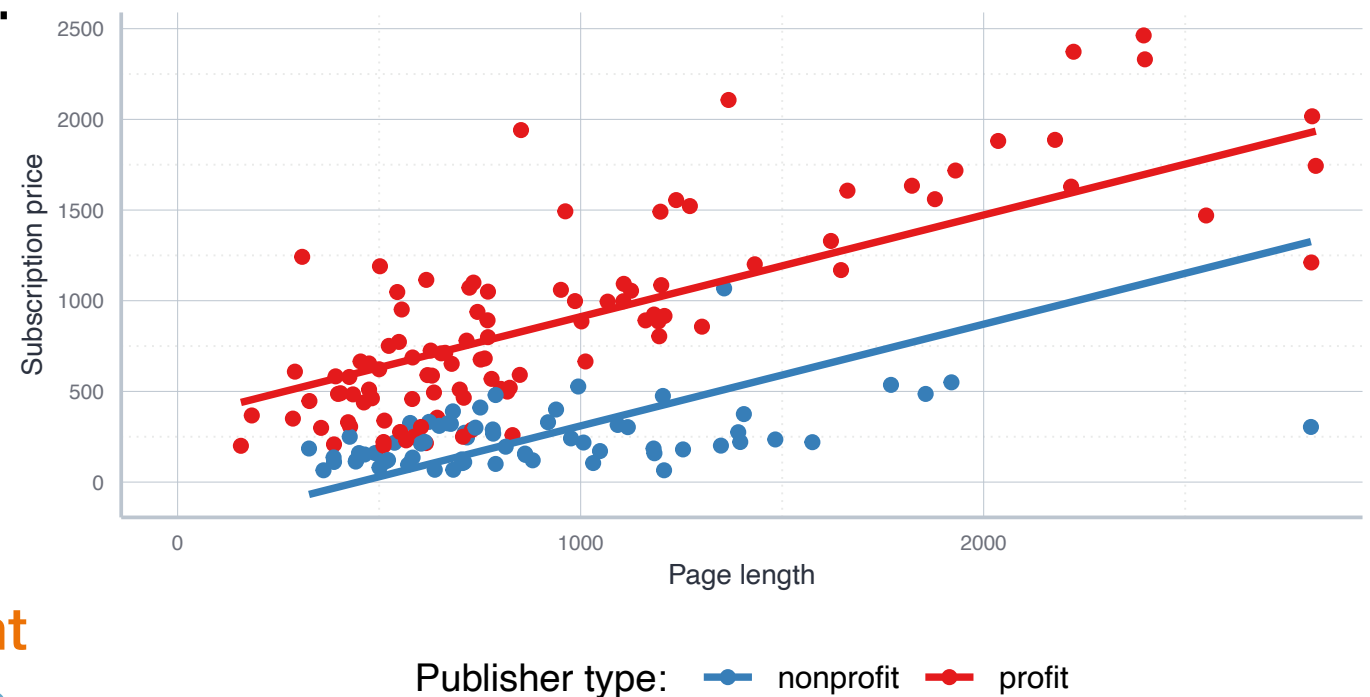
# The parallel slopes model

- The variables `pages_py` and `sub_price` are continuous, the variable `publisher_type` is categorical
- What if we simple add both explanatory variables to the RHS?

`lm(sub_price~papers+publisher_type)`

- Lets look at the resulting estimates:

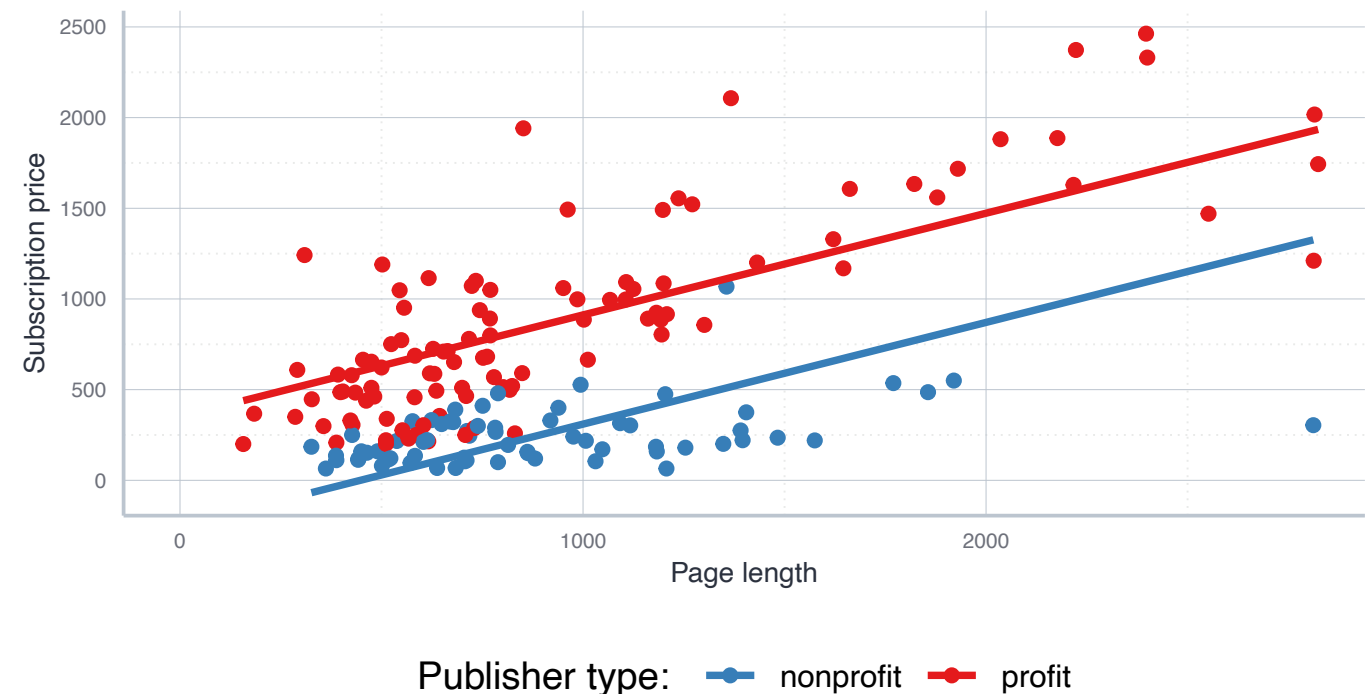
	term	estimate
	<chr>	<dbl>
1	intercept	-251.
2	pages_py	0.561
3	publisher_type: profit	602.



The categorical variables correspond to **different intercepts**, but each group has the **same slope**

# The parallel slopes model

- The results of the parallel slopes model are intuitive in the sense that journals from non-profit publishers are cheaper
- The model suggests, however, that an additional page comes with the same increase in journal price
- The visual inspection, however, indicates that this relationship differs across group



term	estimate
<i>&lt;chr&gt;</i>	<i>&lt;dbl&gt;</i>
1 intercept	-251.
2 pages_py	0.561
3 publisher_type: profit	602.

To capture the idea that the association between page length and price differs across groups we need an **interaction model**

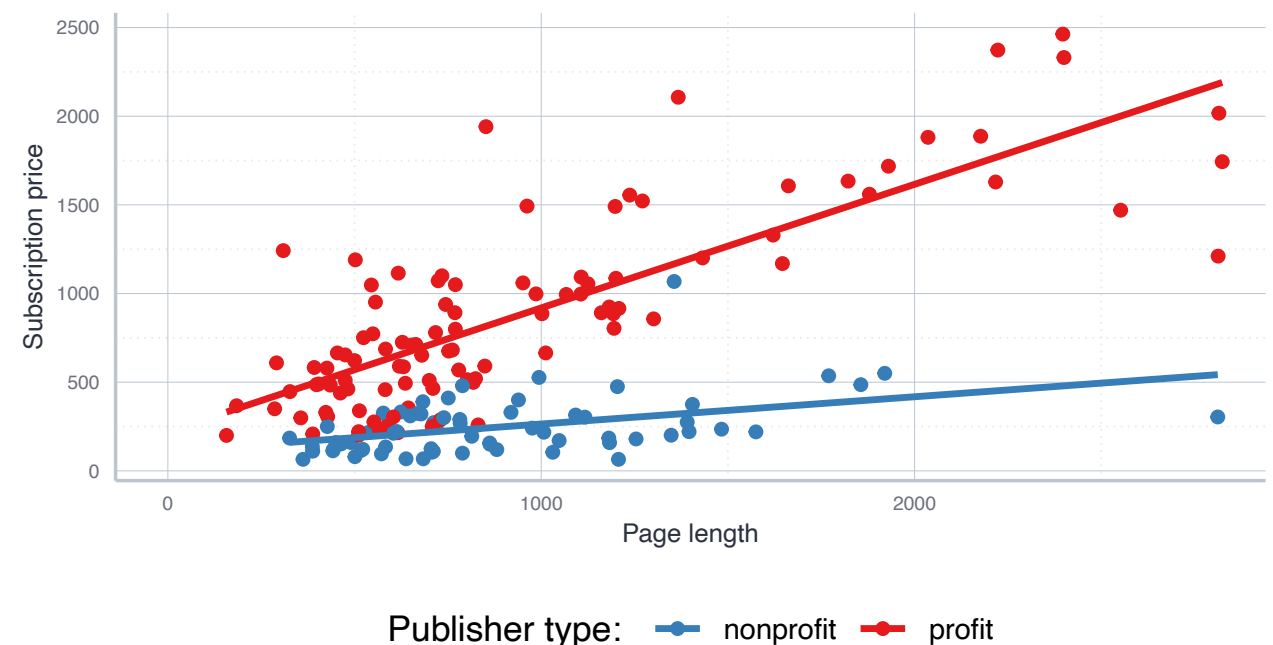
# The interaction model

- This model is more complex: it does not assume that slopes are the same in the different groups → variables interact with each other
- Technically, we just replace the + by an \* in our model formula:

`lm(sub_price~pages_py*publisher_type)`

term	estimate
<i>&lt;chr&gt;</i>	<i>&lt;dbl&gt;</i>
intercept	111.
pages_py	0.154
publisher_type: profit	111.
pages_py:publisher_type:profit	0.543

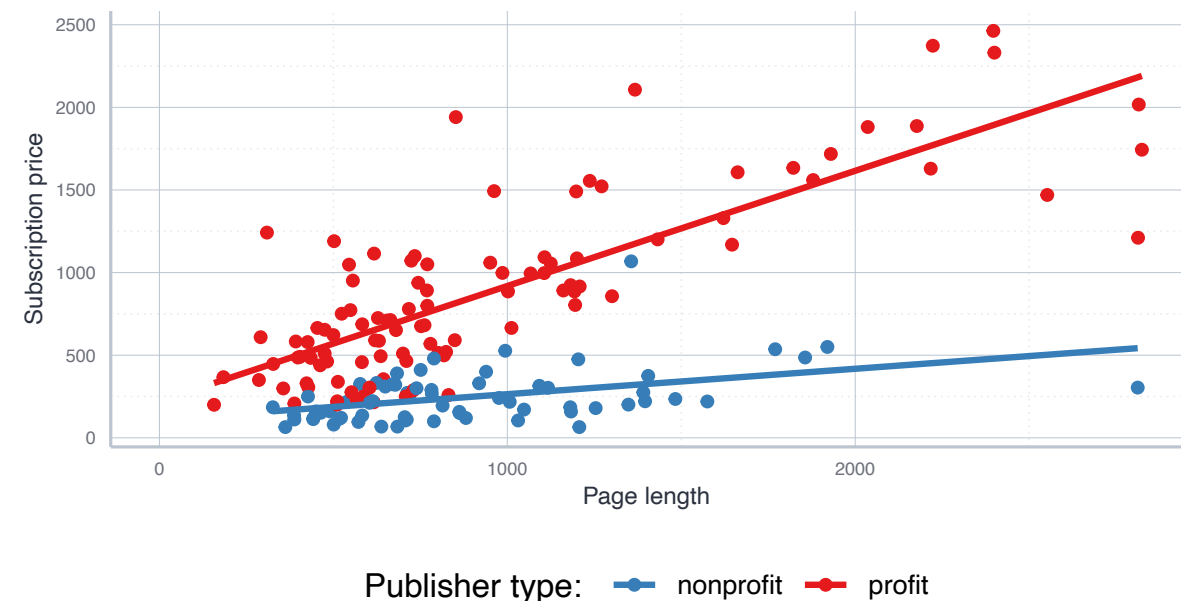
- There is one more parameter to estimate than in the PSM
- But the plot suggests that this additional complexity is warranted: for-profit publisher charge more per additional page



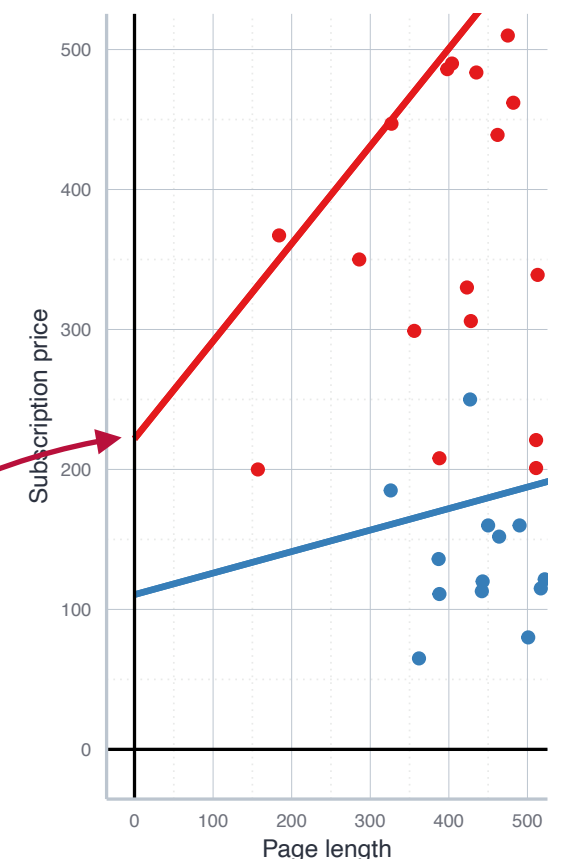
# The interaction model

## Interpretation

term	estimate
<i>&lt;chr&gt;</i>	<i>&lt;dbl&gt;</i>
intercept	111.
pages_py	0.154
publisher_type: profit	111.
pages_py:publisher_typeprofit	0.543



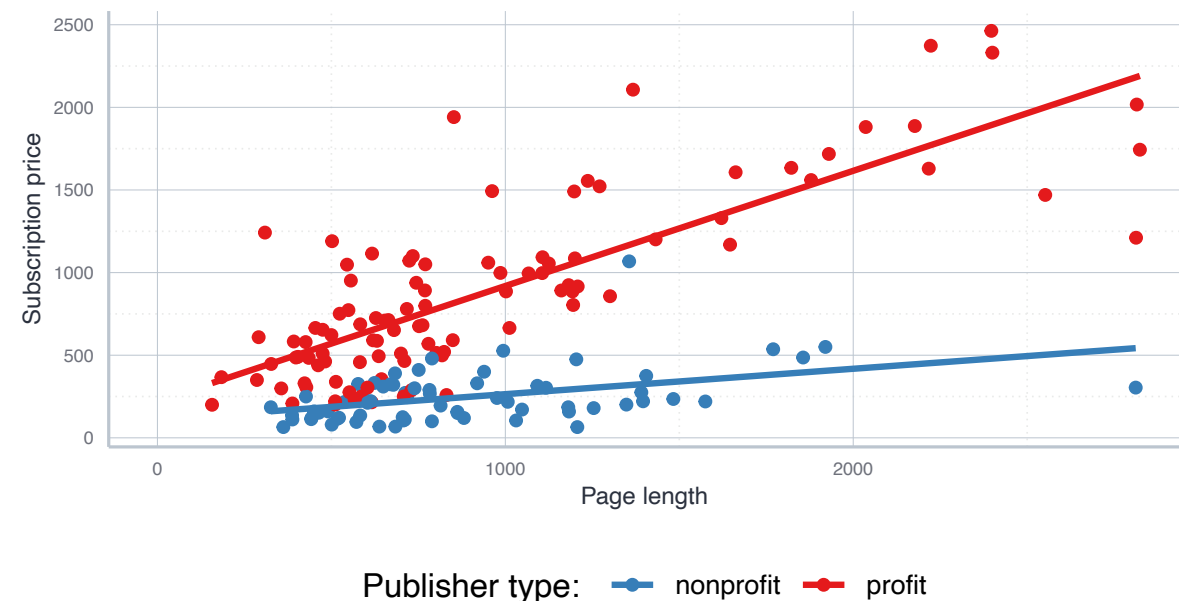
- The estimate `intercept` is the intercept only for the reference group → 111
- The estimate `pages_py` gives the slope only for the reference group → 0.154
- The estimate `publisher_type:profit` gives the difference in the intercept for the profit group
  - $\text{intercept} + \text{publisher\_type:profit} = 222$



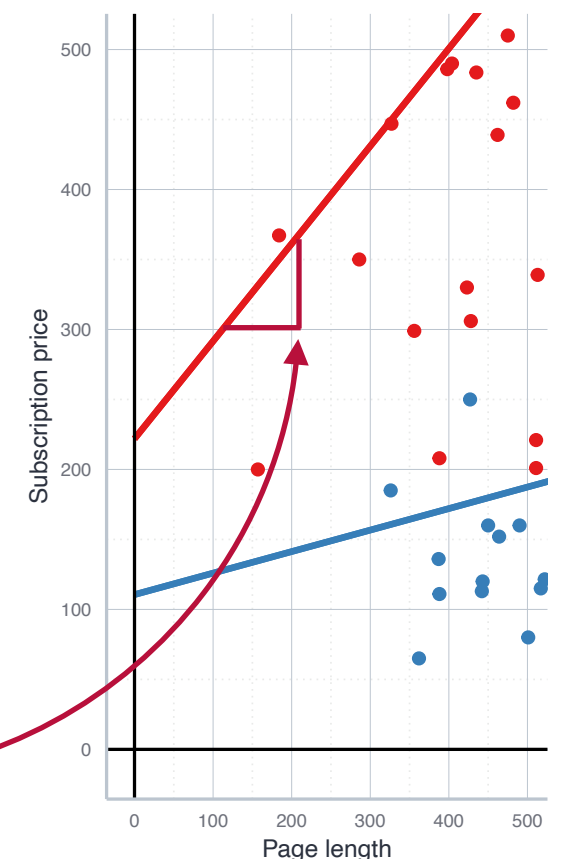
# The interaction model

## Interpretation

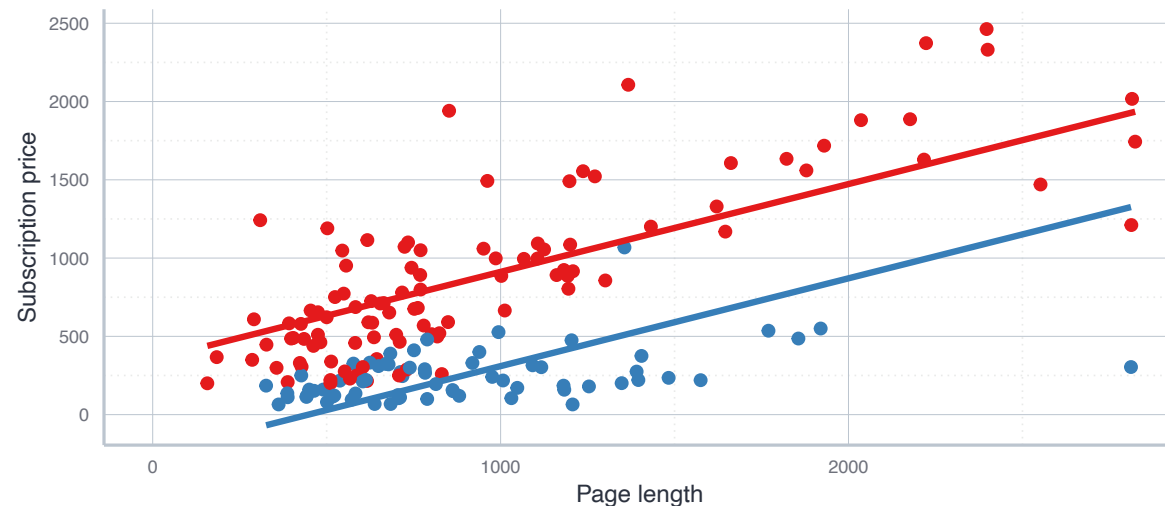
term	estimate
<i>&lt;chr&gt;</i>	<i>&lt;dbl&gt;</i>
intercept	111.
pages_py	0.154
publisher_type: profit	111.
pages_py:publisher_typeprofit	0.543



- The estimate `intercept` is the intercept only for the reference group → 111
- The estimate `pages_py` gives the slope only for the reference group → 0.154
- The estimate `pages_py:profit` gives the difference in the slope for the profit group
  - $\text{pages\_py} + \text{pages\_py:profit} = 0.697$

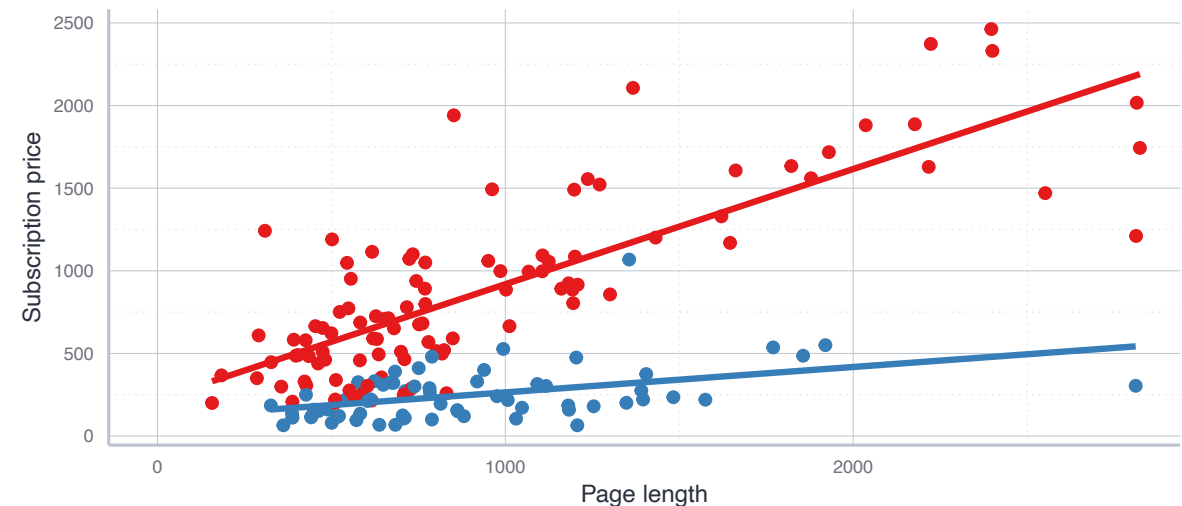


# The interaction and parallel slopes model



Publisher type: — nonprofit — profit

term	estimate
<i>&lt;chr&gt;</i>	<i>&lt;dbl&gt;</i>
1 intercept	-251.
2 pages_py	0.561
3 publisher_type: profit	602.



Publisher type: — nonprofit — profit

term	estimate
<i>&lt;chr&gt;</i>	<i>&lt;dbl&gt;</i>
intercept	111.
pages_py	0.154
publisher_type: profit	111.
pages_py:publisher_type:profit	0.543

- As a general rule of thumb: the PSM is better if nothing suggests that slopes differ → then the estimation is more efficient
- In other cases, its safer to use the interaction mode
- We learn how to test for the right model in later sessions



# Model selection in the multiple variable case

# Model selection using $R^2$

- You can use  $R^2$  as one argument for model selection, i.e. when you need to decide which model works best for your purpose at hand
- Compare, for instance, the  $R^2$  of the PSM and interaction model we estimated before:
- `summary(journal_linmod_intct)[["r.squared"]]: 0.75`
- `summary(journal_linmod_psm)[["r.squared"]]: 0.68`
- The reference to  $R^2$  confirms our impression that the more complex interaction model is warranted
- But: using  $R^2$  in the multiple regression context can be misleading: adding more variables typically increases the  $R^2$  for purely mathematical reasons

# Model selection using $R^2$

- To see why consider the formal definition of  $R^2$ :

$$R^2 = \frac{TSS}{ESS} = \frac{\sum_{i=1}^N (y_i - \bar{y})^2}{\sum_{i=1}^N (\hat{y}_i - \bar{y})^2}$$

- An additional explanatory variable never changes TSS, but mostly increases ESS at least a bit → bias towards 'too complex' models
- There is an alternative, the adjusted  $R^2$ , denoted as  $\bar{R}^2$ :

$$\bar{R}^2 = 1 - \frac{\sum_{i=1}^n e^2 / (N - K - 1)}{\sum_{i=1}^n (Y_i - \bar{Y})^2 / (N - 1)}$$

- Here,  $N$  is the number of observations and  $K$  die Anzahl der zu schätzenden Parameter

# Model selection using $R^2$

- $\bar{R}^2$  only increases if the additional variables contribute to the explanatory power for substantial reasons
  - Drawback: we cannot interpret its value as the share of explained variation any more
- As we learn later, both  $R^2$  and  $\bar{R}^2$  provide valuable information, but they should be complemented by other diagnostic tools
- In the present PSM vs. IM case, using  $\bar{R}^2$  instead of  $R^2$  does not alter the conclusion, but you find plenty other examples in the readings

# Summary & outlook

# Summary

- We extended the simple to the multiple regression model
- This allows us to have more than one variable on the RHS
- The interpretation of the estimates is different:
  - For every increase of 1 unit in the explanatory variable  $i$ , there is an associated decrease of, on average **and ceteris paribus**, of  $\hat{\beta}_i$  units in the response
  - Ceteris paribus: holding all other variables constant
- This allows us to separate the variation in the response variable according to the different explanatory variables
- Forgetting relevant explanatory variables seems to cause problems since adding a variable changes estimates of all other variables

# Summary

- We also learned about how to include categorical variables to regressions
- Technically this is easy, but the interpretation becomes a bit trickier
- When both continuous and categorical variables are used, we learn about the difference between interaction and parallel slope models
- The latter are simpler, but often the complexity of the former model is warranted
- Finally, we saw that selecting models using  $R^2$  requires a bit more caution in the multiple regression context

# Outlook

- Next session we lay the ground for a more nuanced analysis of our model by introducing **sampling theory**
  - How to make statement about populations from which we draw samples?
  - Whenever we want to make such generalisations or predict something with our model, sampling theory is key!

## Tasks until next week:

1. Fill in the **quick feedback survey** on Moodle
2. Read the **tutorials** posted on the course page
3. Do the **exercises** provided on the course page and **discuss problems** and difficulties via the Moodle forum